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# Lower bounds for the approximation with variation-diminishing splines

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**Abstract** We prove lower bounds for the approximation error of the variation-diminishing Schoenberg operator on the interval  $[0, 1]$  in terms of classical moduli of smoothness depending on the degree of the spline basis using a functional analysis based framework. Thereby, we characterize the spectrum of the Schoenberg operator and investigate the asymptotic behavior of its iterates. Finally, we prove the equivalence between the approximation error and the classical second order modulus of smoothness as an improved version of an open conjecture from 2002.

**Keywords** Schoenberg operator · inverse theorem · iterates · spectral theory

## 1 Introduction

Schoenberg introduced the variation-diminishing splines already in 1946 as part of a natural extension of the classical Bernstein polynomials. Even though, his ideas have first been published 20 years later in the well known article of Curry and Schoenberg [4]. Since then, they have attracted the interest of the scientific community due to their good properties and vast range of applications. An comprehensive overview on the theory of splines can be found in the books of de Boor [6], Nürnberger [14], and Schumaker [17].

In 2002, L. Beutel and her coauthors gave in the article [2] a short survey on the history, developments and contributions in this area and investigated quantitative direct approximation inequalities for the Schoenberg operator.

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More importantly, the authors stated an interesting conjecture regarding the equivalence of the approximation error of the Schoenberg operator on  $[0, 1]$  and the second order Ditzian-Totik modulus of smoothness.

We show an even stronger result, namely the equivalence with the classical second order modulus of smoothness. Thereby, we first characterize the asymptotic behavior of the iterates of the Schoenberg operator. Afterwards, we use this result in order to prove a lower bound of the approximation error with respect to the second order modulus of smoothness.

The convergence of the iterates of the Schoenberg operator to the operator of linear interpolation at the endpoints of the interval  $[0, 1]$  can be also seen by the results of the article of Gavrea and Ivan [8]. However, while their methods ensure the uniform convergence of those iterates, they do not give the rate of convergence in which in fact we are interested. Therefore, our approach uses an earlier result of C. Badea [1], where the asymptotic behavior of the iterates is characterized by their spectral properties. Moreover, these results provide a simple and elegant generalization of the results of Nagler and Kähler [13] to the non-uniform case by using a functional analysis based framework, which we believe is beautiful on its own.

### 1.1 The Schoenberg operator

Let  $n > 0, k > 0$  be integers and let  $\Delta_n = \{x_j\}_{j=-k}^{n+k}$  be a partition of  $[0, 1]$  satisfying

$$x_{-k} = \dots = x_0 = 0 < x_1 < \dots < x_n < \dots = x_{n+k} = 1.$$

Throughout this paper, we will consider the Banach space  $C([0, 1])$ , the space of real-valued continuous functions on the interval  $[0, 1]$  endowed with the uniform norm  $\|\cdot\|_\infty$ ,

$$\|f\|_\infty = \sup \{|f(x)| : x \in [0, 1]\}, \quad f \in C([0, 1]).$$

The variation-diminishing spline operator  $S_{\Delta_n, k} : C([0, 1]) \rightarrow C([0, 1])$  of degree  $k$  with respect to the knot sequence  $\Delta_n$  is defined for  $f \in C([0, 1])$  by

$$S_{\Delta_n, k} f(x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x), \quad 0 \leq x < 1,$$

$$S_{\Delta_n, k} f(1) = \lim_{y \nearrow 1} S_{\Delta_n, k} f(y)$$

with the so called Greville nodes

$$\xi_{j,k} := \frac{x_{j+1} + \dots + x_{j+k}}{k}, \quad -k \leq j \leq n-1,$$

and the normalized B-splines

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, \dots, x_{j+k+1}](\cdot - x)_+^k.$$

Here, the divided difference  $[x_j, \dots, x_{j+k+1}]f$  for  $f \in C([0, 1])$  is defined to be the coefficient of  $x^k$  in the unique polynomial of degree  $k$  or less that interpolates the function  $f$  at the points  $x_j, \dots, x_{j+k+1}$ . We denote by  $x_+^k$  the truncated power function of degree  $k$ , defined for  $x \in \mathbb{R}$  by

$$x_+^k = \begin{cases} x^k, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

The operator  $S_{\Delta_n, k}$  was first discussed by Schoenberg and Curry in 1966 as a generalization of the Bernstein operator see, e.g., [4, 12]. The normalized B-splines form a partition of unity

$$\sum_{j=-k}^{n-1} N_{j,k}(x) = 1, \quad (1)$$

and the Schoenberg operator reproduces linear functions, i.e.,

$$\sum_{j=-k}^{n-1} \xi_{j,k} N_{j,k}(x) = x, \quad (2)$$

due to the chosen Greville nodes. A comprehensive overview of direct approximation inequalities for this operator can be found in [2].

## 1.2 Notation

We denote the space of bounded linear operators on  $C([0, 1])$  by  $\mathcal{B}(C([0, 1]))$  equipped with the usual operator norm  $\|\cdot\|_{op}$ . With  $I$  we denote the identity operator on  $\mathcal{B}(C([0, 1]))$ . For  $T \in \mathcal{B}(C([0, 1]))$ , we denote by  $\sigma(T)$  the spectrum of  $T$ ,

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

By  $\sigma_p(T)$ , we denote the point spectrum of  $T$ ,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\},$$

which contains all the eigenvalues of  $T$ . We denote by  $\mathcal{S}(\Delta_n, k)$  the spline space of degree  $k$  with respect to the knot sequence  $\Delta_n$ ,

$$\mathcal{S}(\Delta_n, k) = \left\{ \sum_{j=-k}^{n-1} c_j N_{j,k} : c_j \in \mathbb{R}, j \in \{-k, \dots, n-1\} \right\} \subset C^{k-1}([0, 1]).$$

Since  $\mathcal{S}(\Delta_n, k)$  is an  $n + k$ -dimensional subspace of  $C([0, 1])$ ,  $\mathcal{S}(\Delta_n, k)$  is a Banach space with the inherited norm  $\|\cdot\|_\infty$ . For more information on spline spaces see, e.g., [6, 14, 17]. The open ball of radius  $r > 0$  at the point  $z \in \mathbb{C}$  in the complex plane will be denoted by  $B(z, r) := \{\lambda \in \mathbb{C} : |\lambda - z| < r\}$  and its closure by  $\overline{B}(z, r)$ .

## 2 The spectrum of the Schoenberg operator

We investigate some basic properties of the Schoenberg operator needed in order to prove our main results, and that are of interest of their own. The following fact can, e.g., be found in [5].

**Theorem 1** *The Schoenberg operator  $S_{\Delta_n, k} : C([0, 1]) \rightarrow C([0, 1])$  is bounded and  $\|S_{\Delta_n, k}\|_{op} = 1$ .*

*Proof* Let  $f \in C([0, 1])$  with  $\|f\|_\infty = 1$ . Then

$$\|S_{\Delta_n, k} f\|_\infty = \left\| \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x) \right\|_\infty \leq \|f\|_\infty \cdot \left\| \sum_{j=-k}^{n-1} N_{j,k}(x) \right\|_\infty = 1,$$

because of property (1). Therefore,  $\|S_{\Delta_n, k}\| \leq 1$ . By considering now the constant function  $1 \in C([0, 1])$ , we get  $\|S_{\Delta_n, k} 1\|_\infty = 1$ . Hence, the operator has norm 1,  $\|S_{\Delta_n, k}\|_{op} = 1$ .

Due to the finite-dimensional image of  $S_{\Delta_n, k}$ , we can directly obtain the compactness of the Schoenberg operator.

**Theorem 2** *The Schoenberg operator  $S_{\Delta_n, k} : C([0, 1]) \rightarrow C([0, 1])$  is compact and as such  $\text{Im}(S_{\Delta_n, k} - I)$  is closed. Besides, 1 is not a cluster point of the spectrum  $\sigma(S_{\Delta_n, k})$ .*

*Proof* From Theorem 1 it follows that the operator is bounded with  $\|S_{\Delta_n, k}\|_{op} = 1$  and maps continuous functions to the spline space  $\mathcal{S}(\Delta_n, k)$ . Therefore, the operator has finite rank and finite rank operators are compact. For compact operators  $\text{Im}(T - I)$  is closed and 0 is the only possible cluster point of  $\sigma(S_{\Delta_n, k})$ , see [15], Theorem 4.25.

The main result of this section is the following:

**Theorem 3** *The spectrum of the Schoenberg operator consists only of the point spectrum and*

$$\sigma(S_{\Delta_n, k}) \subset B(0, 1) \cup \{1\}.$$

*Proof* Since  $\|S_{\Delta_n, k}\|_{op} = 1$ , for  $\lambda \in \sigma(S_{\Delta_n, k})$  the inequality

$$|\lambda| \leq \|S_{\Delta_n, k}\|_{op} = 1$$

holds. Therefore,  $\sigma(S_{\Delta_n, k}) \subset \overline{B(0, 1)}$ .

In the following, we show that  $\sigma(S_{\Delta_n, k}) \subset B(0, 1) \cup \{1\}$ , i.e., if  $\lambda \in \sigma(S_{\Delta_n, k})$  with  $|\lambda| = 1$  then  $\lambda = 1$ . First, we prove that  $0 \in \sigma_p(S_{\Delta_n, k})$ . Then, we will show that  $1 \in \sigma_p(S_{\Delta_n, k})$ . Finally, we consider eigenvalues  $\lambda \in \sigma_p(S_{\Delta_n, k}) \setminus \{0, 1\}$  and we show that then  $|\lambda| < 1$  holds.

*Step 1:* We show that  $0 \in \sigma_p(S_{\Delta_{n,k}})$ . Let  $f \in C([0, 1])$  a function, such that

$$f(\xi_j) = 0 \quad \text{for all } j \in \{-k, \dots, n-1\}$$

and such that there exists  $x \in [0, 1] \setminus \{\xi_j : j \in \{-k, \dots, n-1\}\}$  with  $f(x) \neq 0$ . For example, consider the polynomial  $f(x) = \prod_{i=-k}^{n-1} (x - \xi_i)$ . Clearly,  $f \in C([0, 1])$  and we obtain  $S_{\Delta_{n,k}} f = 0 \cdot f = 0$ , because for all  $x \in [0, 1]$

$$S_{\Delta_{n,k}} f(x) = \sum_{j=-k}^{n-1} \left[ \prod_{i=-k}^{n-1} (\xi_j - \xi_i) \right] N_{j,k}(x) = 0.$$

For compact operators, it is known that every  $\lambda \neq 0$  in the spectrum is contained in the point spectrum of the operator. This classical result is stated, e.g., in Theorem 4.25 by Rudin [15].

As  $0 \in \sigma_p(S_{\Delta_{n,k}})$ , it follows that

$$\sigma(S_{\Delta_{n,k}}) = \sigma_p(S_{\Delta_{n,k}}).$$

*Step 2:* We have  $1 \in \sigma(S_{\Delta_{n,k}})$ , because of the properties (1) and (2) and the function  $f(x) = 1$  and  $f(x) = x$  are eigenfunctions of  $S_{\Delta_{n,k}}$  corresponding to the eigenvalue 1.

*Step 3:* Now we prove that for all the other eigenvalues  $\lambda \in \sigma(S_{\Delta_{n,k}})$ , we have

$$|\lambda| < 1.$$

Let  $\lambda \in \sigma(S_{\Delta_{n,k}}) \setminus \{0\}$ . As the operator maps continuous functions to the spline space, the eigenfunctions have to be spline functions as well. Let  $s \in \mathcal{S}(\Delta_{n,k})$ ,  $s = \sum_{j=-k}^{n-1} c_j N_{j,k}$ , be such an eigenfunction for the eigenvalue  $\lambda \in \mathbb{C}$ . Then

$$\begin{aligned} S_{\Delta_{n,k}} s &= \lambda s \\ \iff \sum_{i=-k}^{n-1} \sum_{j=-k}^{n-1} c_j N_{j,k}(\xi_i) N_{i,k}(x) &= \lambda \sum_{i=-k}^{n-1} c_i N_{i,k}(x) \\ \iff \sum_{i=-k}^{n-1} \left[ \sum_{j=-k}^{n-1} c_j N_{j,k}(\xi_i) - \lambda c_i \right] N_{i,k}(x) &= 0 \\ \iff \sum_{j=-k}^{n-1} c_j N_{j,k}(\xi_i) &= \lambda c_i. \end{aligned}$$

Thus,  $\lambda \neq 0$  is an eigenvalue of the operator  $S_{\Delta_{n,k}}$ , if and only if  $\lambda$  is an eigenvalue of the matrix  $N \in \mathbb{R}^{(n+k) \times (n+k)}$ ,

$$N = \begin{pmatrix} N_{-k}(\xi_{-k}) & N_{1-k}(\xi_{-k}) & \cdots & N_{n-1}(\xi_{-k}) \\ N_{-k}(\xi_{1-k}) & N_{1-k}(\xi_{1-k}) & \cdots & N_{n-1}(\xi_{1-k}) \\ \vdots & \vdots & \ddots & \vdots \\ N_{-k}(\xi_{n-1}) & N_{1-k}(\xi_{n-1}) & \cdots & N_{n-1}(\xi_{n-1}) \end{pmatrix}.$$

This matrix  $N$  is non-negative as  $N_j \geq 0$  and every row sums up to one because of property (1). By the Theorem of Gershgorin [9], we have that the eigenvalues are contained in the union of circles

$$\lambda \in \bigcup_{j=-k}^{n-1} D_j,$$

with

$$D_j = \left\{ \lambda \in \mathbb{C} : |\lambda - N_{j,k}(\xi_j)| \leq \sum_{i=-k, i \neq j}^{n-1} N_{j,k}(\xi_i) \right\}.$$

Using property (1), it follows that

$$\bigcup_{j=-k}^{n-1} D_j \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \{1\}.$$

Finally, we obtain  $\sigma_p(S_{\Delta_n,k}) = \sigma(S_{\Delta_n,k}) \subset B(0,1) \cup \{1\}$ .

### 3 Main Results

We investigate the iterates  $S_{\Delta_n,k}^m$  of the Schoenberg operator for  $m \rightarrow \infty$  and prove a lower bound.

#### 3.1 The limit of the iterates of the Schoenberg operator

We show that the iterates of the Schoenberg operator converge in the limit to a linear operator  $L$ . Concretely, we define the iterates by  $S_{\Delta_n,k}^0 = I$  and for  $m \in \mathbb{N}$  by

$$S_{\Delta_n,k}^m f(x) = S_{\Delta_n,k}^{m-1}(S_{\Delta_n,k} f)(x).$$

We will show

$$\lim_{m \rightarrow \infty} \|S_{\Delta_n,k}^m - L\|_{op} = 0,$$

where  $L$  is defined for  $f \in C([0,1])$  by

$$(Lf)(x) = f(0) + (f(1) - f(0)) \cdot x.$$

In [1] it has been shown that operators of a certain structure converge to this linear operator  $L$ . In fact, the Schoenberg operator  $S_{\Delta_n,k} : C([0,1]) \rightarrow C([0,1])$  fulfills the following required properties:

- The operator  $S_{\Delta_n,k}$  is bounded and  $\text{Im}(S_{\Delta_n,k} - I)$  is closed,
- $\ker(S_{\Delta_n,k} - I) = \text{span}(1, x)$ , i.e., the Schoenberg operator reproduces constant and linear functions,
- $S_{\Delta_n,k} f(0) = f(0)$  and  $S_{\Delta_n,k} f(1) = f(1)$  for every  $f \in C([0,1])$ , i.e., the Schoenberg operator interpolates start and end points,

- $\sigma(S_{\Delta_{n,k}}) \subset B(0, 1) \cup \{1\}$ , and finally,
- 1 is not a cluster point<sup>1</sup> of  $\sigma(S_{\Delta_{n,k}})$ , i.e.,

$$\sup \{|\lambda| : \lambda \in \sigma(S_{\Delta_{n,k}}) \setminus \{1\}\} < 1.$$

All these properties were deduced in the previous section. We can conclude:

**Theorem 4** *With  $\gamma_{\Delta_{n,k}} := \sup \{\lambda \in \mathbb{C} : \lambda \in \sigma(S_{\Delta_{n,k}}) \setminus \{1\}\}$ , we obtain*

$$\|S_{\Delta_{n,k}}^m - L\|_{op} \leq C \cdot \gamma_{\Delta_{n,k}}^m$$

*for some suitable constant  $1 \leq C \leq 1/(\gamma_{\Delta_{n,k}})$  and therefore,*

$$\lim_{m \rightarrow \infty} \|S_{\Delta_{n,k}}^m - L\|_{op} = 0.$$

*Proof* The result follows now immediately from Theorem 2.1 in [1] using the above mentioned properties of  $S_{\Delta_{n,k}}$ .

### 3.2 A lower bound of the Schoenberg operator

In this section, we show that for  $r \in \mathbb{N}$ ,  $r \geq 2$ ,  $0 < t \leq \frac{1}{r}$  and  $k > r$ , there exists a constant  $M > 0$ , such that

$$M \cdot \omega_r(f, t) \leq \|f - S_{\Delta_{n,k}} f\|_{\infty}.$$

Here the  $r$ -th modulus of smoothness  $\omega_r : C([0, 1]) \times (0, \frac{1}{r}] \rightarrow [0, \infty)$  is defined by

$$\omega_r(f, t) := \sup_{0 < h < t} \sup \{|\Delta_h^r f(x)| : x \in [0, 1 - rh]\},$$

with the forward difference operator

$$\Delta_h^k f(x) = \sum_{l=0}^k (-1)^{r-l} \binom{r}{l} f(x + lh).$$

The  $r$ -th modulus of smoothness satisfies the following properties [18, 20]:

**Lemma 1** *Let  $0 < t \leq \frac{1}{r}$  be fixed.*

1. *For  $f_1, f_2 \in C([0, 1])$ , the triangle inequality holds,*

$$\omega_r(f_1 + f_2, t) \leq \omega_r(f_1, t) + \omega_r(f_2, t). \quad (3)$$

2. *If  $f \in C([0, 1])$ , then*

$$\omega_r(f, t) \leq 2^r \|f\|_{\infty}. \quad (4)$$

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<sup>1</sup> This condition was not contained in the work of Badea [1], but was needed in our proof for the convergence of the iterates. To the best of our knowledge it is an open question whether this condition is also necessary in the proof for general continuous linear operators as stated in Theorem 2.1 and Theorem 2.2 in [1]. Anyhow, both Theorems hold true for compact operators as in our case.

3. If  $f \in C^r([0, 1])$ , then

$$\omega_r(f, t) \leq t^r \|D^r f\|_\infty. \quad (5)$$

Note that for  $k > r$  the spline space  $\mathcal{S}(\Delta_n, k) \subset C^r([0, 1])$ , because  $S_{\Delta_n, k} f \in C^{k-1}([0, 1])$ . Hence, using inequalities (3) – (5), we obtain

$$\omega_r(f, t) \leq 2^r \|f - S_{\Delta_n, k} f\|_\infty + t^r \|D^r S_{\Delta_n, k} f\|_\infty. \quad (6)$$

See also [3, 10] for the equivalence to the  $K$ -functional. In the following we will estimate the last term with respect to the approximation error  $\|S_{\Delta_n, k} f - f\|_\infty$ . To this end, we consider the minimal mesh length  $\delta_{\min}$ ,

$$\delta_{\min} := \min \{(x_{j+1} - x_j) : j \in \{-k, \dots, n-2\}\}.$$

**Lemma 2** *The differential operator  $D : \mathcal{S}(\Delta_n, k) \rightarrow \mathcal{S}(\Delta_n, k-1)$  is bounded with  $\|D\|_{op} \leq (2/\delta_{\min})d_k$ , where  $d_k > 0$  is a constant depending only on  $k$ .*

*Proof* Let  $s \in \mathcal{S}(\Delta_n, k)$ ,  $s(x) = \sum_{j=-k}^{n-1} c_j N_{j,k} x$ , with  $\|s\|_\infty = 1$ . According to [12], we can calculate the derivative by

$$Ds(x) = \sum_{j=1-k}^{n-1} \frac{c_j - c_{j-1}}{\xi_j - \xi_{j-1}} N_{j,k-1}(x).$$

Then we obtain with the triangle inequality

$$\begin{aligned} \|Ds\|_\infty &= \left\| \sum_{j=1-k}^{n-1} \frac{c_j - c_{j-1}}{\xi_j - \xi_{j-1}} N_{j,k-1} \right\|_\infty \\ &\leq \frac{\|c\|_\infty + \|c\|_\infty}{\delta_{\min}} \cdot \left\| \sum_{j=1-k}^{n-1} N_{j,k-1} \right\|_\infty, \end{aligned}$$

where

$$\|c\|_\infty = \max \{|c_j| : j \in \{-k, \dots, n-1\}\}. \quad (7)$$

According to [5], there exists  $d_k > 0$  depending only on  $k$ , such that

$$d_k^{-1} \|c\|_\infty \leq \left\| \sum_{j=-k}^{n-1} c_j N_{j,k} \right\|_\infty \leq \|c\|_\infty. \quad (8)$$

Rewriting the first inequality yields  $\|c\|_\infty \leq d_k$ , because  $\|s\|_\infty = 1$ . Now we use the partition of the unity (1) to derive the estimate

$$\|Ds\|_\infty \leq \frac{2}{\delta_{\min}} d_k.$$

Taking the supremum of all  $s \in \mathcal{S}(\Delta_n, k)$  with  $\|s\|_\infty = 1$  yields the result.



**Corollary 1** *For  $l < k$ , the differential operators  $D^l : \mathcal{S}(\Delta_n, k) \rightarrow \mathcal{S}(\Delta_n, k - l)$  are bounded and*

$$\|D^l\|_{op} \leq (2/\delta_{\min})^l d_k.$$

*Remark 1* The asymptotic behaviour of the constant  $d_k$  in Lemma 2 is already characterized quite well in the literature. C. de Boor conjectured that

$$d_k \sim 2^k$$

holds for all  $k > 0$ . He showed in [7] with numerical computations that

$$d_k \leq c \cdot 2^k$$

for some constant  $c > 0$ . In [11], T. Lyche proved the lower bound

$$2^{-3/2} \frac{k-1}{k} \cdot 2^k \leq d_k.$$

C. de Boor's conjecture was confirmed in the article [16] of Scherer and Shadrin up to a polynomial factor. There the authors showed that the inequality

$$d_k \leq k \cdot 2^k$$

holds for all  $k > 0$ . In our interest is the relation  $d_k \rightarrow \infty$  if  $k$  tends to infinity.

Now we are able to estimate  $\|D^r S_{\Delta_n, k} f\|_\infty$  in terms of the approximation error  $\|f - S_{\Delta_n, k} f\|_\infty$ .

**Theorem 5** *Let  $f \in C([0, 1])$ ,  $r \geq 2$ ,  $k > r$  and  $0 < t \leq \frac{1}{r}$ . Then there exists  $M > 0$ , such that*

$$M \cdot \omega_r(f, t) \leq \|f - S_{\Delta_n, k} f\|_\infty.$$

*Proof* We derive

$$\begin{aligned} \|D^r S_{\Delta_n, k} f\|_\infty &= \|D^r S_{\Delta_n, k} f - D^r S_{\Delta_n, k}^2 f + D^r S_{\Delta_n, k}^2 f - D^r S_{\Delta_n, k}^3 f + \dots\|_\infty \\ &\leq \sum_{m=1}^{\infty} \|D^r S_{\Delta_n, k}^m (f - S_{\Delta_n, k} f)\|_\infty \\ &\leq \|f - S_{\Delta_n, k} f\|_\infty \sum_{m=1}^{\infty} \|D^r S_{\Delta_n, k}^m\|_{op} \\ &= \|f - S_{\Delta_n, k} f\|_\infty \sum_{m=1}^{\infty} \|D^r (S_{\Delta_n, k}^m - L + L)\|_{op} \\ &= \|f - S_{\Delta_n, k} f\|_\infty \sum_{m=1}^{\infty} \|D^r (S_{\Delta_n, k}^m - L)\|_{op}, \end{aligned}$$

as  $D^r$  annihilates linear functions and therefore,  $D^r L = 0$ . Then we obtain using Theorem 4 and Corollary 1

$$\|D^r S_{\Delta_n, k} f\| \leq \|f - S_{\Delta_n, k} f\|_\infty \|D^r\|_{op} \sum_{m=1}^{\infty} \|S_{\Delta_n, k}^m - L\|_{op}$$

$$\begin{aligned}
&\leq \|f - S_{\Delta_n, k} f\|_\infty \|D^r\|_{op} \sum_{m=1}^{\infty} C \gamma_{\Delta_n, k}^m \\
&\leq \|f - S_{\Delta_n, k} f\|_\infty \|D^r\|_{op} \frac{C \gamma_{\Delta_n, k}}{1 - \gamma_{\Delta_n, k}} \\
&\leq \frac{2^r \gamma_{\Delta_n, k} d_k C}{\delta_{\min}^r (1 - \gamma_{\Delta_n, k})} \|f - S_{\Delta_n, k} f\|_\infty.
\end{aligned}$$

As  $C \leq 1/\gamma_{\Delta_n, k}$ , we get

$$\|D^r S_{\Delta_n, k} f\|_\infty \leq \frac{2^r d_k}{\delta_{\min}^r (1 - \gamma_{\Delta_n, k})} \|f - S_{\Delta_n, k} f\|_\infty.$$

Applying inequality (6) for  $0 < t \leq \frac{1}{r}$  yields

$$\omega_r(f, t) \leq 2^r \left( 1 + \frac{d_k}{\delta_{\min}^r (1 - \gamma_{\Delta_n, k})} t^r \right) \cdot \|f - S_{\Delta_n, k} f\|_\infty.$$

**Corollary 2** *For all  $f \in C([0, 1])$  and  $r \geq 2$ , the approximation error cannot be better than*

$$\frac{1}{2^{r+1}} \omega_r(f, \delta) \leq \|f - S_{\Delta_n, k} f\|_\infty,$$

where

$$\delta = \delta_{\min} \cdot \left( \frac{1 - \gamma_{\Delta_n, k}}{d_k} \right)^{1/r}$$

given a fixed grid  $\Delta_n$  and the degree  $k$  of the spline approximation. For  $n \rightarrow \infty$  and  $k \rightarrow \infty$ , we have  $\delta \rightarrow 0$  as  $\delta_{\min} \rightarrow 0$  and  $d_k \rightarrow \infty$  respectively.

*Remark 2* Note that our result can also be generalized to the Ditzian-Totik modulus of smoothness, as a similar equivalence to a weighted K-functional holds.

**Corollary 3** *For  $0 < t \leq \frac{1}{2}$  and  $k > 2$ , we obtain the equivalence*

$$\omega_2(f, t) \sim \|f - S_{\Delta_n, k} f\|_\infty$$

in the sense that there exist constants  $M_1, M_2 > 0$  independent of  $f$  and only depending on  $\Delta_n$  and  $k$  such that

$$M_1 \cdot \omega_r(f, t) \leq \|f - S_{\Delta_n, k} f\|_\infty \leq M_2 \cdot \omega_r(f, t).$$

*Proof* We apply Corollary 2 to get the lower inequality

$$\frac{1}{8} \cdot \omega_2(f, \sqrt{\frac{(1 - \gamma_{\Delta_n, k}) \cdot \delta_{\min}^2}{d_k}}) \leq \|f - S_{\Delta_n, k} f\|_\infty.$$

We use the inequality

$$\|f - S_{\Delta_n, k} f\|_\infty \leq \frac{3}{2} \cdot \omega_2(f, \sqrt{\min \left\{ \frac{1}{2k}, \frac{(k+1) \cdot \delta_{\max}^2}{12} \right\}})$$

from [2], Theorem 6, to obtain the upper bound. Here

$$\delta_{\max} := \max \{ (x_{j+1} - x_j) : j \in \{-k, \dots, n-1\} \}.$$

Finally, there is still one open question to answer. By definition of the constants, we have  $d_k \rightarrow \infty$  for  $k \rightarrow \infty$  and  $\delta_{\min} \rightarrow 0$  for  $n \rightarrow \infty$ . The question is whether the second largest eigenvalues of the operator can speed up the convergence in Corollary 2. As far as we know, the eigenvalues and eigenfunctions of the Schoenberg operator are still unknown. We conclude the article with the following conjecture that characterizes the behavior of the second largest eigenvalue of the Schoenberg operator.

*Conjecture 1* Let  $k > 0$  be fixed. Then

$$\gamma_{\Delta_n, k} \rightarrow 1, \quad \text{for } n \rightarrow \infty.$$

Let  $n > 0$  be fixed. Then

$$\gamma_{\Delta_n, k} \rightarrow 1, \quad \text{for } k \rightarrow \infty.$$

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